

POSITIVE FINITE RANK ELEMENTARY OPERATORS AND CHARACTERIZING ENTANGLEMENT OF STATES

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ABSTRACT. In this paper, a class of indecomposable positive finite rank elementary operators of order (n, n) are constructed. This allows us to give a simple necessary and sufficient criterion for separability of pure states in bipartite systems of any dimension in terms of positive elementary operators of order $(2, 2)$ and get some new mixed entangled states that can not be detected by the positive partial transpose (PPT) criterion and the realignment criterion.

1. INTRODUCTION

Entanglement is a basic physical resource to realize various quantum information and quantum communication tasks such as quantum cryptography, teleportation, dense coding and key distribution [14]. Let H and K be separable complex Hilbert spaces. Recall that a quantum state is a density operator $\rho \in \mathcal{B}(H \otimes K)$ which is positive and has trace 1. Denote by $\mathcal{S}(H)$ the set of all states on $H \otimes K$. If H and K are finite dimensional, $\rho \in \mathcal{S}(H \otimes K)$ is said to be separable if ρ can be written as

$$\rho = \sum_{i=1}^k p_i \rho_i \otimes \sigma_i,$$

where ρ_i and σ_i are states on H and K respectively, and p_i are positive numbers with $\sum_{i=1}^k p_i = 1$. Otherwise, ρ is said to be inseparable or entangled (ref. [1, 14]). For the case that at least one of H and K is of infinite dimension, by Werner [16], a state ρ acting on $H \otimes K$ is called separable if it can be approximated in the trace norm by the states of the form

$$\sigma = \sum_{i=1}^n p_i \rho_i \otimes \sigma_i,$$

where ρ_i and σ_i are states on H and K respectively, and p_i are positive numbers with $\sum_{i=1}^n p_i = 1$. Otherwise, ρ is called an entangled state.

PACS. 03.65.Ud, 03.65.Db, 03.67.-a.

Key words and phrases. Quantum states, entanglement, positive linear maps.

This work is partially supported by National Natural Science Foundation of China (No. 10771157), Research Grant to Returned Scholars of Shanxi (2007-38) and Foundation of Shanxi University.

It is very important but also difficult to determine whether or not a state in a composite system is separable. For 2×2 and 2×3 systems, that is, for the case $\dim H = \dim K = 2$ or $\dim H = 2$, $\dim K = 3$, a state is separable if and only if it is a positive partial transpose (PPT) state (see [2, 3]), but it has no efficiency for PPT entangled states appearing in the higher dimensional systems. In [4], the realignment criterion for separability in finite-dimensional systems was found. A most general approach to characterize quantum entanglement is based on the notion of entanglement witnesses (see [2]). A Hermitian operator W acting on $H \otimes K$ is said to be an entanglement witness (briefly, EW), if W is not positive and $\text{Tr}(W\sigma) \geq 0$ holds for all separable states σ . Thus, if W is an EW, then there exists an entangled state ρ such that $\text{Tr}(W\rho) < 0$ (that is, the entanglement of ρ can be detected by W). It was shown that, a state is entangled if and only if it is detected by some entanglement witnesses [2]. However, constructing entanglement witnesses is a hard task. There was a considerable effort in constructing and analyzing the structure of entanglement witnesses for finite and infinite dimensional systems [5, 6, 7, 8, 12] (see also [9] for a review). Recently, Hou and Qi in [12] showed that every entangled state in a bipartite system can be detected by some entanglement witness W of the form $W = cI + T$ with I the identity operator, c a nonnegative number and T a finite rank self-adjoint operator.

It is obvious that if ρ is a state on $H \otimes K$, then for every completely positive linear map $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$, the operator $(\Phi \otimes I)\rho \in \mathcal{B}(K \otimes K)$ is always positive; if ρ is separable then for every positive linear map $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$, the operator $(\Phi \otimes I)\rho$ is always positive on $K \otimes K$ (or, for every positive linear map $\Phi : \mathcal{B}(K) \rightarrow \mathcal{B}(H)$, the operator $(I \otimes \Phi)\rho$ is always positive on $H \otimes H$). For the finite dimensional cases, the converse of the last statement is also true since, due to the Choi-Jamiołkowski isomorphism, any EW on $H \otimes K$ corresponds to a positive linear map $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$. In [2], the following positive map criterion was established.

Horodeckis' Theorem. ([2, Theorem 2]) *Let H, K be finite dimensional complex Hilbert spaces and ρ be a state acting on $H \otimes K$. Then ρ is separable if and only if for any positive linear map $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$, the operator $(\Phi \otimes I)\rho$ is positive on $K \otimes K$.*

Recently, Hou in [13] improved the above result and established the elementary operator criterion for infinite dimensional bipartite systems.

The elementary operator criterion. ([13, Theorem 4.5]) *Let H, K be complex Hilbert spaces and ρ be a state acting on $H \otimes K$. Then the following statements are equivalent.*

- (1) ρ is separable;
- (2) $(\Phi \otimes I)\rho \geq 0$ holds for every positive elementary operator $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$.

(3) $(\Phi \otimes I)\rho \geq 0$ holds for every positive finite rank elementary operator $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$.

Therefore, a state ρ on $H \otimes K$ is entangled if and only if there exists a positive finite rank elementary operator $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ which is not completely positive (briefly, NCP) such that $(\Phi \otimes I)\rho$ is not positive. Thus it is very important and interesting to find as many as possible positive finite rank elementary operators that are NCP, and then, to apply them to detect the entanglement of states.

The purpose of this paper is to construct some new indecomposable positive finite rank elementary operators and apply them to get some new examples of entangled states that can not be detected by the PPT criterion and the realignment criterion. Recall that, a positive map Δ is said to be decomposable if it is the sum of a completely positive map Δ_1 and the composition of a completely positive map Δ_2 and the transpose \mathbf{T} , i.e., $\Delta = \Delta_1 + \mathbf{T} \circ \Delta_2$.

The paper is organized as follows. Section 2 is devoted to giving some preliminary results on characterizing positive elementary operators and introducing a concept of the order of finite rank elementary operators (Definition 2.8). In Section 3, we give a simple necessary and sufficient condition for a pure state to be separable in bipartite systems of any dimension in terms of a special positive elementary operator of order $(2, 2)$ (Theorem 3.1). Then we use a class of positive finite rank elementary operators of order $(3, 3)$ (see Theorem 3.2) to detect entanglement of some states (Example 3.3 and 3.4). The purpose of Section 4 is to obtain a new class of positive finite rank elementary operators of order $(4, 4)$ that are not decomposable (Theorem 4.1), and then, apply them to detect the entanglement of two kinds of states (Example 4.2 and 4.3). Some new examples of PPT entangled states that can not be detected by the realignment criterion are obtained. Section 5 is devoted to discussing the general case. For any $n \geq 3$, a new class of indecomposable positive finite rank elementary operators of order (n, n) are constructed (Theorem 5.1) and the entanglement of two kinds of states are detected (Examples 5.4-5.5). In Section 6, a short conclusion is given.

Throughout this paper, H and K are complex Hilbert spaces of any dimension, and $\langle \cdot | \cdot \rangle$ stands for the inner product in both of them. $\mathcal{B}(H, K)$ ($\mathcal{B}(H)$ when $K = H$) is the Banach space of all (bounded linear) operators from H into K . $A \in \mathcal{B}(H)$ is self-adjoint if $A = A^\dagger$ (A^\dagger stands for the adjoint operator of A); and A is positive, denoted by $A \geq 0$, if $\langle \psi | A | \psi \rangle \geq 0$ for all $|\psi\rangle \in H$. For any positive integer n , $H^{(n)}$ denotes the direct sum of n copies of H . It is clear that every operator $\mathbf{A} \in \mathcal{B}(H^{(n)}, K^{(m)})$ can be written in an $m \times n$ operator matrix $\mathbf{A} = (A_{ij})_{i,j}$ with $A_{ij} \in \mathcal{B}(H, K)$, $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$. Equivalently, $\mathcal{B}(H^{(n)}, K^{(m)})$ is often written as $\mathcal{B}(H, K) \otimes \mathcal{M}_{m \times n}(\mathbb{C})$. If Φ is a linear map from $\mathcal{B}(H)$ into $\mathcal{B}(K)$, we can define a linear map $\Phi_n : \mathcal{B}(H^{(n)}) \rightarrow \mathcal{B}(K^{(n)})$ by $\Phi_n((A_{ij})) = (\Phi(A_{ij}))$. Recall that Φ is positive

if $A \in \mathcal{B}(H)$ is positive implies that $\Phi(A)$ is positive; Φ is n -positive if Φ_n is positive; Φ is completely positive if Φ_n is positive for every integer $n > 0$. A linear map $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ is called an elementary operator if there are two finite sequences $\{A_i\}_{i=1}^n \subset \mathcal{B}(H, K)$ and $\{B_i\}_{i=1}^n \subset \mathcal{B}(K, H)$ such that $\Phi(X) = \sum_{i=1}^n A_i X B_i$ for all $X \in \mathcal{B}(H)$. It was shown in [10] that an elementary operator Φ is of finite rank if and only if there exist finite rank operators $A_i, B_i, i = 1, 2, \dots, k$, such that $\Phi(X) = \sum_{i=1}^k A_i X B_i$.

2. A CHARACTERIZATION OF POSITIVE ELEMENTARY OPERATORS

In this section, we give some preliminary results on characterizing positive elementary operators, which are needed in this paper.

Before stating the main results in this section, let us recall some notions from [11]. Let $l, k \in \mathbb{N}$ (the set of all natural numbers), and let A_1, \dots, A_k , and $C_1, \dots, C_l \in \mathcal{B}(H, K)$. If, for each $|\psi\rangle \in H^{(m)}$ (the direct sum of m copies of H), there exists an $l \times k$ complex matrix $(\alpha_{ij}(|\psi\rangle))$ (depending on $|\psi\rangle$) such that

$$C_i^{(m)}|\psi\rangle = \sum_{j=1}^k \alpha_{ij}(|\psi\rangle) A_j^{(m)}|\psi\rangle, \quad i = 1, 2, \dots, l,$$

we say that (C_1, \dots, C_l) is an m -locally linear combination of (A_1, \dots, A_k) , $(\alpha_{ij}(|\psi\rangle))$ is called a *local coefficient matrix* at $|\psi\rangle$. Furthermore, if a local coefficient matrix $(\alpha_{ij}(|\psi\rangle))$ can be chosen for every $|\psi\rangle \in H^{(m)}$ so that its operator norm $\|(\alpha_{ij}(|\psi\rangle))\| \leq 1$, we say that (C_1, \dots, C_l) is an m -contractive locally linear combination of (A_1, \dots, A_k) ; if there is a matrix (α_{ij}) such that $C_i = \sum_{j=1}^k \alpha_{ij} A_j$ for all i , we say that (C_1, \dots, C_l) is a *linear combination* of (A_1, \dots, A_k) with coefficient matrix (α_{ij}) . We'll omit " m " in the case $m = 1$. Sometimes we also write $\{A_i\}_{i=1}^k$ for (A_1, \dots, A_k) .

The following characterization of m -positive elementary operators was obtained in [11], also, see [13]. If $m = 1$, we get a characterization of positive elementary operators.

Theorem 2.1. *Let $\Phi(\cdot) = \sum_{i=1}^n A_i(\cdot)B_i$ be an elementary operator from $\mathcal{B}(H)$ into $\mathcal{B}(K)$. Φ is m -positive if and only if there exist C_1, \dots, C_k and D_1, \dots, D_l in $\text{span}\{A_1, \dots, A_n\}$ with $k+l \leq n$ such that (D_1, \dots, D_l) is an m -contractive locally linear combination of (C_1, \dots, C_k) and*

$$\Phi(X) = \sum_{i=1}^k C_i(X)C_i^\dagger - \sum_{j=1}^l D_j(X)D_j^\dagger \quad (2.1)$$

for all $X \in \mathcal{B}(H)$. Furthermore, Φ in Eq.(2.1) is completely positive if and only if (D_1, \dots, D_l) is a linear combination of (C_1, \dots, C_k) with a contractive coefficient matrix, and in turn, if

and only if there exist E_1, E_2, \dots, E_r with $r \leq k$ such that

$$\Phi = \sum_{i=1}^r E_i(\cdot) E_i^\dagger.$$

It is obvious that if $\Phi(\cdot) = \sum_{i=1}^n A_i(\cdot) B_i$ sends self-adjoint operators to self-adjoint operators, then Φ can be represented in the form $\Phi(\cdot) = \sum_{i=1}^k C_i(\cdot) C_i^\dagger - \sum_{j=1}^l D_j(\cdot) D_j^\dagger$ with C_1, \dots, C_k and D_1, \dots, D_l in $\text{span}\{A_1, \dots, A_n\}$. Furthermore, Theorem 2.1 says that, Φ is m -positive if and only if (D_1, \dots, D_l) is an m -contractive locally linear combination of (C_1, \dots, C_k) , and Φ is completely positive if and only if (D_1, \dots, D_l) is a contractive linear combination of (C_1, \dots, C_k) .

Since every linear map between matrix algebras is an elementary operator, by Theorem 2.1, we have

Corollary 2.2. *Let H and K be finite dimensional complex Hilbert spaces and let $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ be a linear map. Then Φ is positive if and only if there exist $C_1, \dots, C_k, D_1, \dots, D_l \in \mathcal{B}(H, K)$ such that $\{D_j\}_{j=1}^l$ is a contractive locally linear combination of $\{C_i\}_{i=1}^k$ and $\Phi(X) = \sum_{i=1}^k C_i X C_i^\dagger - \sum_{j=1}^l D_j X D_j^\dagger$ for all $X \in \mathcal{B}(H)$.*

If $\mathcal{L} \subset \mathcal{B}(H, K)$, we will denote by \mathcal{L}_F the subset of all finite-rank operators in \mathcal{L} .

By Theorem 2.1, we can get some useful simple conditions to ensure that a positive elementary operator is completely positive or not. The Corollaries 2.3-2.5 below can be found in [11, 13].

Corollary 2.3. *Assume that $\Phi = \sum_{i=1}^k A_i(\cdot) A_i^\dagger - \sum_{j=1}^l B_j(\cdot) B_j^\dagger : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ is a positive elementary operator. If any one of the following conditions holds, then Φ is completely positive:*

- (i) $k \leq 2$.
- (ii) $\dim(\text{span}\{A_1, \dots, A_k\}_F) \leq 2$.
- (iii) *There exists a vector $|\psi\rangle \in H$ such that $\{A_i|\psi\rangle\}_{i=1}^k$ is linearly independent.*
- (iv) Φ is $[\frac{k+1}{2}]$ -positive, where $[t]$ stands for the integer part of the real number t .

Corollary 2.4. *Assume that $\Phi = \sum_{i=1}^k A_i(\cdot) A_i^\dagger - \sum_{j=1}^l B_j(\cdot) B_j^\dagger : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ is a positive elementary operator. If Φ is not completely positive, then*

- (i) $k \geq 3$,
- (ii) $\dim(\text{span}\{A_1, \dots, A_k\}_F) \geq 3$,
- (iii) *For every vector $|\psi\rangle \in H$, $\{A_i|\psi\rangle\}_{i=1}^k$ is linearly dependent.*
- (iv) B_j is a finite-rank perturbation of some combination of $\{A_i\}_{i=1}^k$ for each $j = 1, 2, \dots, l$.
- (v) $\Phi_{[\frac{k+1}{2}]}$ is not positive.

Corollary 2.5. Assume that $\Phi = \sum_{i=1}^k A_i(\cdot)A_i^\dagger - \sum_{j=1}^l B_j(\cdot)B_j^\dagger : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ is an elementary operator. If there exists some j such that B_j is not a contractive linear combination of $\{A_i\}_{i=1}^k$, then Φ is not completely positive.

The following result is easily checked and useful to us.

Proposition 2.6. Let

$$B_{(t_1, t_2, \dots, t_n)} = \begin{pmatrix} t_1 & -1 & -1 & \cdots & -1 \\ -1 & t_2 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & t_n \end{pmatrix} \in M_n(\mathbb{C}).$$

If $t_i \geq n-1$ for each $i = 1, 2, \dots, n$, then $B_{(t_1, t_2, \dots, t_n)} \geq 0$ (that is, $B_{(t_1, t_2, \dots, t_n)}$ is semi-positive definite); if $t_i < n-1$ for each $i = 1, 2, \dots, n$, then $B_{(t_1, t_2, \dots, t_n)} \not\geq 0$. Particularly, $B_{(t, \dots, t)} \geq 0$ if and only if $t \geq n-1$.

Proof. Assume that $t_i \geq n-1$ for each $i = 1, 2, \dots, n$. Then $t_0 = \min\{t_1, t_2, \dots, t_n\} \geq n-1$. For any $|x\rangle = (\xi_1, \xi_2, \dots, \xi_n)^T \in \mathbb{C}^n$, we have

$$\begin{aligned} \langle x | B_{(t_1, t_2, \dots, t_n)} | x \rangle &= t_0 \sum_{i=1}^n |\xi_i|^2 - 2 \sum_{i < j} \xi_i \bar{\xi}_j \\ &\geq t_0 \sum_{i=1}^n |\xi_i|^2 - 2 \sum_{i < j} |\xi_i| |\xi_j| \\ &= (t_0 - n + 1) \sum_{i=1}^n |\xi_i|^2 + (n-1) \sum_{i=1}^n |\xi_i|^2 - 2 \sum_{i < j} |\xi_i| |\xi_j| \\ &= (t_0 - n + 1) \sum_{i=1}^n |\xi_i|^2 + \sum_{i < j} (|\xi_i| - |\xi_j|)^2 \geq 0. \end{aligned}$$

which implies that $B_{(t_1, t_2, \dots, t_n)} \geq 0$. If $t_i < n-1$ for each $i = 1, 2, \dots, n$, then $t'_0 = \max\{t_1, t_2, \dots, t_n\} < n-1$. Taking $\xi_1 = \xi_2 = \dots = \xi_n \neq 0$ and let $|x_0\rangle = (\xi_1, \xi_1, \dots, \xi_1)^T$, one gets $\langle x_0 | B_{(t_1, t_2, \dots, t_n)} | x_0 \rangle \leq (t'_0 - n + 1)n \sum_{i=1}^n |\xi_1|^2 < 0$. It follows that $B_t \not\geq 0$, completing the proof. \square

By using of above results, we can prove the following result.

Proposition 2.7. Let H and K be Hilbert spaces and let $\{|i\rangle\}_{i=1}^n$ and $\{|i'\rangle\}_{i=1}^n$ be any orthonormal sets of H and K , respectively. Denote $E_{ji} = |j'\rangle\langle i| \in \mathcal{B}(H, K)$. Let $\Delta : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ be defined by

$$\Delta_{(t_1, t_2, \dots, t_n)}(A) = \sum_{i=1}^n t_i E_{ii} A E_{ii}^\dagger - (\sum_{i=1}^n E_{ii}) A (\sum_{i=1}^n E_{ii})^\dagger$$

for all $A \in \mathcal{B}(H)$. If $t_i \geq n$ for each $i = 1, 2, \dots, n$, then $\Delta_{(t_1, t_2, \dots, t_n)}$ is a completely positive map; if $t_i < n$ for each $i = 1, 2, \dots, n$, then $\Delta_{(t_1, t_2, \dots, t_n)}$ is not a positive map. Particularly, $\Delta_{(t, \dots, t)}$ is positive if and only if it is completely positive, and in turn, if and only if $t \geq n$.

Proof. For any unit vector $|x\rangle = (\xi_1, \xi_2, \dots, \xi_n, 0, 0, \dots)^T \in H$, consider the rank-one projection $|x\rangle\langle x|$. We have

$$\Delta(|x\rangle\langle x|) = \left(\begin{array}{cccc|ccc} (t_1 - 1)|\xi_1|^2 & -\xi_1\bar{\xi}_2 & \cdots & -\xi_1\bar{\xi}_n & 0 & 0 & \cdots \\ -\xi_2\bar{\xi}_1 & (t_2 - 1)|\xi_2|^2 & \cdots & -\xi_2\bar{\xi}_n & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\xi_n\bar{\xi}_1 & -\xi_n\bar{\xi}_2 & \cdots & (t_n - 1)|\xi_n|^2 & 0 & 0 & \cdots \\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{array} \right). \quad (2.2)$$

If $t_i < n$ for each $i = 1, 2, \dots, n$, taking $|x\rangle = (1, 1, \dots, 1, 0, 0, \dots)^T$ in Eq.(2.2) and by Proposition 2.6, we get $\Delta(|x\rangle\langle x|) \not\geq 0$, and so Δ is not positive.

On the other hand, assume that $t_i \geq n$ for each $i = 1, 2, \dots, n$. Since $\sum_{i=1}^n E_{ii} = \sum_{i=1}^n \frac{1}{\sqrt{t_i}}(\sqrt{t_i}E_{ii})$ and $\sum_{i=1}^n (\frac{1}{\sqrt{t_i}})^2 \leq \sum_{i=1}^n (\frac{1}{\sqrt{n}})^2 \leq 1$, $\sum_{i=1}^n E_{ii}$ is a contractive linear combination of $\{\sqrt{t_1}E_{11}, \sqrt{t_2}E_{22}, \dots, \sqrt{t_n}E_{nn}\}$. By Theorem 2.1, Δ is completely positive. \square

For the sake of convenience, we introduce a terminology here.

Definition 2.8. Let $\Delta : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ be a finite frank elementary operator. It follows from a characterization of finite rank elementary operators in [10] that there exist finite rank projections $P \in \mathcal{B}(H)$ and $Q \in \mathcal{B}(K)$ such that

$$\Delta(A) = Q\Delta(PAP)Q \text{ for all } A \in \mathcal{B}(H). \quad (2.3)$$

Let

$$(n, m) = \min\{(\text{rank}(P), \text{rank}(Q)) : (P, Q) \text{ satisfies the equation (2.3)}\}.$$

(n, m) is called the order of Δ , and we say that the elementary operator Δ is of the order (n, m) .

3. POSITIVE FINITE RANK ELEMENTARY OPERATORS OF ORDER (2, 2) AND (3, 3)

In this section we will construct some positive finite rank elementary operators of order (2, 2) and (3, 3). Applying such positive maps, we give a simple necessary and sufficient condition for a pure state to be separable. We also use these positive maps to detect some entangled mixed states.

Positive elementary operators of order (2, 2) are easily constructed. For example, Let H and K be Hilbert spaces of dimension ≥ 2 , and let $\{|i\rangle\}_{i=1}^{\dim H}$ and $\{|j'\rangle\}_{j=1}^{\dim K}$ be any orthonormal

sets of H and K , respectively. Let $\Phi_0 : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ be defined by

$$\begin{aligned} \Phi_0(A) = & E_{11}AE_{11}^\dagger + E_{22}AE_{22}^\dagger + E_{12}AE_{12}^\dagger \\ & + E_{21}AE_{21}^\dagger - (E_{11} + E_{22})A(E_{11} + E_{22})^\dagger \end{aligned} \quad (3.1)$$

and

$$\begin{aligned} \Psi_0(A) = & (2E_{11} + E_{22})A(2E_{11} + E_{22})^\dagger + E_{12}AE_{12}^\dagger \\ & + E_{21}AE_{21}^\dagger - (E_{11} + E_{22})A(E_{11} + E_{22})^\dagger \end{aligned} \quad (3.2)$$

for every $A \in \mathcal{B}(H)$, where $E_{ji} = |j'\rangle\langle i|$. It is obvious that both Φ_0 and Ψ_0 are positive because the map

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

and the map

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} 3a_{11} + a_{22} & a_{12} \\ a_{21} & a_{11} \end{pmatrix}$$

on $M_2(\mathbb{C})$ are positive. A surprising fact is that such simple positive elementary operator of order $(2, 2)$ will be enough to determine the separability of the pure states.

Let $\mathcal{U}(H)$ (resp. $\mathcal{U}(K)$) be the group of all unitary operators on H (resp. on K). For any map $\Delta : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ and any unitary operators $U \in \mathcal{U}(H)$ and $V \in \mathcal{U}(K)$, the deduced map $A \mapsto V^\dagger \Delta(U^\dagger A U) V$ will be denoted by $\Delta^{U,V}$. The next result give a simple necessary and sufficient criterion of separability for pure states in bipartite composite systems of any dimension. This criterion is easily performed.

Theorem 3.1. *Let H and K be Hilbert spaces of dimension ≥ 2 , and let $\{|i\rangle\}_{i=1}^{\dim H}$ and $\{|j'\rangle\}_{j=1}^{\dim K}$ be any orthonormal sets of H and K , respectively. Let $\Phi_0(\Psi_0) : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ be defined by Eq.(3.1) (Eq.(3.2)). Then a pure state ρ on $H \otimes K$ is separable if and only if*

$$(\Phi_0^{U,V} \otimes I)\rho \geq 0 \quad ((\Psi_0^{U,V} \otimes I)\rho \geq 0)$$

holds for all $U \in \mathcal{U}(H)$ and $V \in \mathcal{U}(K)$.

Proof. If a state ρ is separable, then $(\Phi_0^{U,V} \otimes I)\rho \geq 0$ $((\Psi_0^{U,V} \otimes I)\rho \geq 0)$ as $\Phi_0^{U,V}$ ($\Psi_0^{U,V}$) is a positive map.

Conversely, assume that $\rho = |\psi\rangle\langle\psi|$ is an inseparable pure state. Let $|\psi\rangle = \sum_{k=1}^{N_\psi} \delta_k |k, k'\rangle$ be the Schmidt decomposition, where $\delta_1 \geq \delta_2 \geq \dots > 0$ with $\sum_{k=1}^{N_\psi} \delta_k^2 = 1$, and $\{|k\rangle\}_{k=1}^{N_\psi}$ and $\{|k'\rangle\}_{k=1}^{N_\psi}$ are orthonormal in H and K , respectively. Thus $\rho = \sum_{k,l=1}^{N_\psi} \delta_k \delta_{k'} |k, k'\rangle\langle l, l'| = \sum_{k,l=1}^{N_\psi} \delta_k \delta_{k'} E_{kl} \otimes E_{k'l'}$. Since $\rho = |\psi\rangle\langle\psi|$ is inseparable, the Schmidt number N_ψ of $|\psi\rangle$ is greater than 1 and hence $\delta_1 \geq \delta_2 > 0$.

Up to unitary equivalence, we may assume that $\{|k\rangle\}_{k=1}^2 = \{|i\rangle\}_{i=1}^2$ and $\{|k'\rangle\}_{k'=1}^2 = \{|j'\rangle\}_{j'=1}^2$. Then, since $\Phi_0(E_{kl}) = 0$ ($\Psi_0(E_{kl}) = 0$) whenever $k > 2$ or $l > 2$, we have

$$\begin{aligned} (\Phi_0 \otimes I)\rho &= \sum_{i,j=1}^2 \delta_i \delta_j \Phi_0(E_{ij}) \otimes E_{ij} \\ &\cong \begin{pmatrix} 0 & 0 & 0 & -\delta_1 \delta_2 \\ 0 & \delta_1^2 & 0 & 0 \\ 0 & 0 & \delta_2^2 & 0 \\ -\delta_1 \delta_2 & 0 & 0 & 0 \end{pmatrix} \oplus 0 \end{aligned}$$

$$\begin{aligned} ((\Psi_0 \otimes I)\rho &= \sum_{i,j=1}^2 \delta_i \delta_j \Psi_0(E_{ij}) \otimes E_{ij} \\ &\cong \begin{pmatrix} 3\delta_1^2 & 0 & 0 & \delta_1 \delta_2 \\ 0 & \delta_1^2 & 0 & 0 \\ 0 & 0 & \delta_2^2 & 0 \\ \delta_1 \delta_2 & 0 & 0 & 0 \end{pmatrix} \oplus 0), \end{aligned}$$

which is clearly not positive. \square

Now let us consider the positive elementary operators of order $(3, 3)$.

Theorem 3.2. *Let H and K be Hilbert spaces of dimension ≥ 3 , and let $\{|i\rangle\}_{i=1}^3$ and $\{|j'\rangle\}_{j'=1}^3$ be any orthonormal sets of H and K , respectively. Let $\Phi, \Phi' : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ be defined by*

$$\begin{aligned} \Phi(A) &= 2 \sum_{i=1}^3 E_{ii} A E_{ii}^\dagger + E_{12} A E_{12}^\dagger + E_{23} A E_{23}^\dagger + E_{31} A E_{31}^\dagger \\ &\quad - (\sum_{i=1}^3 E_{ii}) A (\sum_{i=1}^3 E_{ii})^\dagger \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} \Phi'(A) &= 2 \sum_{i=1}^3 E_{ii} A E_{ii}^\dagger + E_{13} A E_{13}^\dagger + E_{21} A E_{21}^\dagger + E_{32} A E_{32}^\dagger \\ &\quad - (\sum_{i=1}^3 E_{ii}) A (\sum_{i=1}^3 E_{ii})^\dagger \end{aligned} \quad (3.3)'$$

for every $A \in \mathcal{B}(H)$, where $E_{ji} = |j'\rangle\langle i|$. Then Φ and Φ' are indecomposable positive finite rank elementary operators of order $(3, 3)$.

Proof. We only give the proof that Φ is NCP positive. Φ' is dealt with similarly.

It is obvious that Φ is a finite rank elementary operator of order $(3, 3)$. Also, it is clear from Theorem 2.1 that Φ is not completely positive because $\sum_{i=1}^3 E_{ii}$ is not a contractive linear combination of $\{\sqrt{2}E_{11}, \sqrt{2}E_{22}, \sqrt{2}E_{33}, E_{12}, E_{23}, E_{31}\}$. To prove the positivity of Φ , extend $\{|i\rangle\}_{i=1}^3$ and $\{|j'\rangle\}_{j'=1}^3$ to orthonormal bases $\{|i\rangle\}_{i=1}^{\dim H}$ and $\{|j'\rangle\}_{j'=1}^{\dim K}$ of H and K , respectively. Then every $A \in \mathcal{B}(H)$ has a matrix representation $A = (a_{kl})$ and the map Φ

maps A into

$$\Phi(A) = \begin{pmatrix} a_{11} + a_{22} & -a_{12} & -a_{13} & 0 & 0 & \cdots \\ -a_{21} & a_{22} + a_{33} & -a_{23} & 0 & 0 & \cdots \\ -a_{31} & -a_{32} & a_{33} + a_{11} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which is unitarily equivalent to

$$S \oplus 0 = \begin{pmatrix} a_{11} + a_{22} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} + a_{33} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} + a_{11} \end{pmatrix} \oplus 0.$$

By [13, Proposition 5.2], the matrix S is positive. So $\Phi(A)$ is positive. The fact that Φ is not decomposable will be proved by Example 3.3 or 3.4, completing the proof of the theorem. \square

Next we use the positive maps in Theorem 3.2 to detect some mixed entangled states. These examples also imply that the positive maps in Theorem 3.2 are not decomposable since they can recognize some PPT entangled states.

The states ρ in Example 3.3 was discussed in [8] and their entanglement were detected by constructing suitable witnesses.

Example 3.3. Let H and K be Hilbert spaces and let $\{|i\rangle\}_{i=1}^3$ and $\{|j'\rangle\}_{j=1}^3$ be any orthonormal sets of H and K , respectively. Let $|\omega\rangle = \frac{1}{\sqrt{3}}(|11'\rangle + |22'\rangle + |33'\rangle)$, and define $\rho_1 = |\omega\rangle\langle\omega|$, $\rho_2 = \frac{1}{3}(|12'\rangle\langle 12'| + |23'\rangle\langle 23'| + |31'\rangle\langle 31'|)$ and $\rho_3 = \frac{1}{3}(|13'\rangle\langle 13'| + |21'\rangle\langle 21'| + |32'\rangle\langle 32'|)$. Let $\rho = \sum_{i=1}^3 q_i \rho_i$ and $\rho_t = (1-t)\rho + t\rho_0$, where $q_i \geq 0$ for $i = 1, 2, 3$ with $q_1 + q_2 + q_3 = 1$, $t \in [0, 1]$, and ρ_0 is a state on $H \otimes K$. By the positive finite rank elementary operators Φ and Φ' defined by Eq.(3.3) and Eq.(3.3)', respectively, we obtain that, for sufficiently small t or for any ρ_0 with $(\Phi \otimes I)\rho_0 = (\Phi' \otimes I)\rho_0 = 0$, the following statements are true.

- (1) If $q_i < q_1$ for some $i = 2, 3$, then ρ_t is entangled.
- (2) Let ρ_0 be PPT. Then ρ_t is PPT if and only if $q_i q_j \geq q_1^2$. Thus, if $0 < q_i < q_1 < \frac{1}{3}$ and $\frac{1}{3} < q_j < 1$ with $q_i q_j \geq q_1^2$, where $i, j \in \{2, 3\}$ and $i \neq j$, then ρ_t is PPT entangled.

In fact, by [12], we need only to check the following:

- (1)' if $q_i < q_1$ for some $i = 2, 3$, then ρ is entangled;
- (2)' ρ is PPT if and only if $q_i q_j \geq q_1^2$. Thus, if $0 < q_i < q_1 < \frac{1}{3}$ and $\frac{1}{3} < q_j < 1$ with $q_i q_j \geq q_1^2$, where $i, j \in \{2, 3\}$ and $i \neq j$, then ρ is PPT entangled.

For the map Φ , we have

$$\rho = \frac{1}{3} \begin{pmatrix} q_1 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & q_1 \\ 0 & q_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_2 & 0 & 0 & 0 & 0 & 0 \\ q_1 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 0 & 0 & q_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_2 & 0 \\ q_1 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & q_1 \end{pmatrix} \oplus 0$$

and

$$\begin{aligned} & 3(\Phi \otimes I)(\rho) \\ & \cong \begin{pmatrix} q_1 + q_3 & 0 & 0 & 0 & -q_1 & 0 & 0 & 0 & -q_1 \\ 0 & q_2 + q_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_1 + q_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_1 + q_2 & 0 & 0 & 0 & 0 & 0 \\ -q_1 & 0 & 0 & 0 & q_1 + q_3 & 0 & 0 & 0 & -q_1 \\ 0 & 0 & 0 & 0 & 0 & q_2 + q_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_2 + q_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_1 + q_2 & 0 \\ -q_1 & 0 & 0 & 0 & -q_1 & 0 & 0 & 0 & q_1 + q_3 \end{pmatrix} \oplus 0 \\ & \cong A_1 \oplus B_1 \oplus C_1 \oplus 0, \end{aligned}$$

where \cong means “be unitarily equivalent to”,

$$A_1 = \begin{pmatrix} q_1 + q_3 & -q_1 & -q_1 \\ -q_1 & q_1 + q_3 & -q_1 \\ -q_1 & -q_1 & q_1 + q_3 \end{pmatrix}, \quad B_1 = \begin{pmatrix} q_1 + q_2 & 0 & 0 \\ 0 & q_1 + q_2 & 0 \\ 0 & 0 & q_1 + q_2 \end{pmatrix}$$

and

$$C_1 = \begin{pmatrix} q_2 + q_3 & 0 & 0 \\ 0 & q_2 + q_3 & 0 \\ 0 & 0 & q_2 + q_3 \end{pmatrix}.$$

It is obvious that $B_1, C_1 \geq 0$. For A_1 , by Proposition 2.6, we have $A_1 \not\geq 0$ if $q_3 < q_1$. It follows from the elementary operator criterion that ρ is entangled if $q_3 < q_1$. Moreover, it is easily checked that ρ is PPT if and only if $q_2 q_3 \geq q_1^2$. Thus we obtain that ρ is PPT entangled if $0 < q_3 < q_1 < \frac{1}{3}$ and $\frac{1}{3} \leq q_2 < 1$ with $q_2 q_3 \geq q_1^2$.

Similarly, by using of the map Φ' , one gets the other half of the assertions (1)'-(2)'.

The states ρ_t in the next example were introduced in [12] firstly.

Example 3.4. Let H and K be complex Hilbert spaces and let $\{|i\rangle\}_{i=1}^{\dim H}$ and $\{|j\rangle\}_{j=1}^{\dim K}$ be any orthonormal bases of H and K , respectively. Let

$$|\omega_1\rangle = \frac{1}{\sqrt{3}}(|11'\rangle + |22'\rangle + |33'\rangle) \quad \text{and} \quad |\omega_2\rangle = \frac{1}{\sqrt{3}}(|12'\rangle + |23'\rangle + |31'\rangle).$$

Define $\rho_1 = |\omega_1\rangle\langle\omega_1|$, $\rho_2 = |\omega_2\rangle\langle\omega_2|$ and $\rho_3 = \frac{1}{3}(|13'\rangle\langle 13'| + |21'\rangle\langle 21'| + |32'\rangle\langle 32'|)$. Let $\rho = \sum_{i=1}^3 q_i \rho_i$ and $\rho_t = (1-t)\rho + t\rho_0$, where $q_i \geq 0$ for $i = 1, 2, 3$ with $q_1 + q_2 + q_3 = 1$, $t \in [0, 1]$, and ρ_0 is a state on $H \otimes K$.

Hou and Qi in [12] proved that, if $q_2 < \frac{5}{7}q_1$ or $q_1 < \frac{5}{7}q_2$, then, for sufficient small t , ρ_t is entangled; if $q_2 < \frac{5}{7}q_1$ or $q_1 < \frac{5}{7}q_2$, and if $q_1 q_2 q_3 \geq q_1^3 + q_2^3$, then ρ_t is PPT entangled whenever ρ_0 is. Now, by using of the positive finite rank elementary operators Φ and Φ' in Theorem 3.2, we can give a finer result. In fact, for sufficient small t , or for ρ_0 with $(\Phi \otimes I)\rho_0 = (\Phi' \otimes I)\rho_0 = 0$ (for example, taking $\rho_0 = \sum_{i=4}^{\infty} p_i |i\rangle\langle i'| \otimes |i\rangle\langle i'|$, $p_i \geq 0$, $\sum_{i=4}^{\infty} p_i = 1$), the following statements are true.

- (1) If $q_1 \neq q_2$ or $q_1 = q_2 > q_3$, then ρ_t is entangled;
- (2) Let ρ_0 be PPT. Then ρ_t is PPT if and only if $q_1 q_2 q_3 \geq q_1^3 + q_2^3$. Particularly, if $q_j = 2q_i$ and $\frac{9}{2}q_j \leq q_3$, where $i, j \in \{1, 2\}$ and $i \neq j$, then ρ_t is PPT entangled.

Still, we need only consider ρ and check the following:

- (1)' If $q_1 \neq q_2$ or $q_1 = q_2 > q_3$, then ρ is entangled;
- (2)' ρ is PPT if and only if $q_1 q_2 q_3 \geq q_1^3 + q_2^3$. Particularly, if $q_j = 2q_i$ and $\frac{9}{2}q_j \leq q_3$, where $i, j \in \{1, 2\}$ and $i \neq j$, then ρ is PPT entangled.

For $\rho = q_1 \rho_1 + q_2 \rho_2 + q_3 \rho_3$, it is obvious that

$$\rho = \frac{1}{3} \begin{pmatrix} q_1 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & q_1 \\ 0 & q_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_2 & q_2 & 0 & 0 & 0 & q_2 & 0 \\ 0 & 0 & q_2 & q_2 & 0 & 0 & 0 & q_2 & 0 \\ q_1 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 0 & 0 & q_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_3 & 0 & 0 \\ 0 & 0 & q_2 & q_2 & 0 & 0 & 0 & q_2 & 0 \\ q_1 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & q_1 \end{pmatrix} \oplus 0.$$

Note that

$$\begin{aligned} & 3(\Phi \otimes I)(\rho) \\ & \cong \begin{pmatrix} q_1 + q_3 & 0 & 0 & 0 & -q_1 & 0 & 0 & 0 & -q_1 \\ 0 & q_2 + q_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_1 + q_2 & -q_2 & 0 & 0 & 0 & -q_2 & 0 \\ 0 & 0 & -q_2 & q_1 + q_2 & 0 & 0 & 0 & -q_2 & 0 \\ -q_1 & 0 & 0 & 0 & q_1 + q_3 & 0 & 0 & 0 & -q_1 \\ 0 & 0 & 0 & 0 & 0 & q_2 + q_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_2 + q_3 & 0 & 0 \\ 0 & 0 & -q_2 & -q_2 & 0 & 0 & 0 & q_1 + q_2 & 0 \\ -q_1 & 0 & 0 & 0 & -q_1 & 0 & 0 & 0 & q_1 + q_3 \end{pmatrix} \oplus 0, \end{aligned}$$

which is unitarily equivalent to the operator $A \oplus B \oplus C \oplus 0$, where

$$A = \begin{pmatrix} q_1 + q_3 & -q_1 & -q_1 \\ -q_1 & q_1 + q_3 & -q_1 \\ -q_1 & -q_1 & q_1 + q_3 \end{pmatrix}, \quad B = \begin{pmatrix} q_1 + q_2 & -q_2 & -q_2 \\ -q_2 & q_1 + q_2 & -q_2 \\ -q_2 & -q_2 & q_1 + q_2 \end{pmatrix}$$

and

$$C = \begin{pmatrix} q_2 + q_3 & 0 & 0 \\ 0 & q_2 + q_3 & 0 \\ 0 & 0 & q_2 + q_3 \end{pmatrix} \geq 0.$$

For the matrices A and B , by Proposition 2.6, we get that $A \not\geq 0$ if $q_3 < q_1$ and $B \not\geq 0$ if $q_1 < q_2$. So $(\Phi \otimes I)(\rho)$ is not positive if $q_3 < q_1$ or $q_1 < q_2$. It follows from the elementary operator criterion that ρ is entangled if $q_3 < q_1$ or $q_1 < q_2$. Note that ρ is PPT if and only if $q_1 q_2 q_3 \geq q_1^3 + q_2^3$. Thus particularly we obtain that ρ is PPT entangled if $q_2 = 2q_1$ and $\frac{9}{2}q_1 \leq q_3$.

Similarly, by applying the map Φ' , one can get that the other half of the assertions (1)'-(2)' is true.

4. POSITIVE FINITE RANK ELEMENTARY OPERATORS OF ORDER $(4, 4)$

In this section we construct a class of positive finite rank elementary operators of order $(4, 4)$. The following is our main result.

Theorem 4.1. *Let H and K be Hilbert spaces of dimension greater than 3 and let $\{|i\rangle\}_{i=1}^4$ and $\{|j'\rangle\}_{j=1}^4$ be any orthonormal sets of H and K , respectively. Let $\Phi, \Phi', \Phi'' : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ be defined by*

$$\begin{aligned} \Phi(A) = & 3 \sum_{i=1}^4 E_{ii} A E_{ii}^\dagger + E_{12} A E_{12}^\dagger + E_{23} A E_{23}^\dagger + E_{34} A E_{34}^\dagger + E_{41} A E_{41}^\dagger \\ & - (\sum_{i=1}^4 E_{ii}) A (\sum_{i=1}^4 E_{ii})^\dagger, \end{aligned} \quad (4.1)$$

$$\begin{aligned} \Phi'(A) = & 3 \sum_{i=1}^4 E_{ii} A E_{ii}^\dagger + E_{13} A E_{13}^\dagger + E_{24} A E_{24}^\dagger + E_{31} A E_{31}^\dagger + E_{42} A E_{42}^\dagger \\ & - (\sum_{i=1}^4 E_{ii}) A (\sum_{i=1}^4 E_{ii})^\dagger \end{aligned} \quad (4.1)'$$

and

$$\begin{aligned} \Phi''(A) = & 3 \sum_{i=1}^4 E_{ii} A E_{ii}^\dagger + E_{14} A E_{14}^\dagger + E_{21} A E_{21}^\dagger + E_{32} A E_{32}^\dagger + E_{43} A E_{43}^\dagger \\ & - (\sum_{i=1}^4 E_{ii}) A (\sum_{i=1}^4 E_{ii})^\dagger \end{aligned} \quad (4.1)''$$

for every $A \in \mathcal{B}(H)$, where $E_{ji} = |j'\rangle\langle i|$. Then Φ, Φ', Φ'' are positive finite rank elementary operators that are not completely positive. Moreover, Φ and Φ'' are indecomposable.

Proof. Still, we only prove that Φ is positivity but not completely positive.

It is clear from Theorem 2.1 that Φ is not completely positive because $\sum_{i=1}^4 E_{ii}$ is not a contractive linear combination of $\{\sqrt{3}E_{11}, \dots, \sqrt{3}E_{44}, E_{12}, E_{23}, E_{34}, E_{41}\}$. We will show that Φ is positive. Extend $\{|i\rangle\}_{i=1}^4$ and $\{|j'\rangle\}_{j=1}^4$ to orthonormal bases $\{|i\rangle\}_{i=1}^{\dim H}$ and $\{|j'\rangle\}_{j=1}^{\dim K}$ of H and K , respectively. Then every $A \in \mathcal{B}(H)$ has a matrix representation $A = (a_{kl})$. Obviously, Φ maps $A = (a_{kl})$ to the matrix

$$\Phi(A) = \begin{pmatrix} 2a_{11} + a_{22} & -a_{12} & -a_{13} & -a_{14} & 0 & \cdots \\ -a_{21} & 2a_{22} + a_{33} & -a_{23} & -a_{24} & 0 & \cdots \\ -a_{31} & -a_{32} & 2a_{33} + a_{44} & -a_{34} & 0 & \cdots \\ -a_{41} & -a_{42} & -a_{43} & 2a_{44} + a_{11} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Take any unit vector $|x\rangle = (x_1, x_2, x_3, x_4, x_5, \dots)^T \in H$ and consider the rank-one projection $|x\rangle\langle x|$. Obviously, Φ is positive if and only if $\Phi(|x\rangle\langle x|) \geq 0$ holds for all unit vector $x \in H$.

Since

$$\Phi(|x\rangle\langle x|) = \begin{pmatrix} 2|x_1|^2 + |x_2|^2 & -x_1\bar{x}_2 & -x_1\bar{x}_3 & -x_1\bar{x}_4 & 0 & \cdots \\ -x_2\bar{x}_1 & 2|x_2|^2 + |x_3|^2 & -x_2\bar{x}_3 & -x_2\bar{x}_4 & 0 & \cdots \\ -x_3\bar{x}_1 & -x_3\bar{x}_2 & 2|x_3|^2 + |x_4|^2 & -x_3\bar{x}_4 & 0 & \cdots \\ -x_4\bar{x}_1 & -x_4\bar{x}_2 & -x_4\bar{x}_3 & 2|x_4|^2 + |x_1|^2 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

we see that $\Phi(|x\rangle\langle x|) \geq 0$ if and only if

$$M(x) = \begin{pmatrix} 2|x_1|^2 + |x_2|^2 & -x_1\bar{x}_2 & -x_1\bar{x}_3 & -x_1\bar{x}_4 \\ -x_2\bar{x}_1 & 2|x_2|^2 + |x_3|^2 & -x_2\bar{x}_3 & -x_2\bar{x}_4 \\ -x_3\bar{x}_1 & -x_3\bar{x}_2 & 2|x_3|^2 + |x_4|^2 & -x_3\bar{x}_4 \\ -x_4\bar{x}_1 & -x_4\bar{x}_2 & -x_4\bar{x}_3 & 2|x_4|^2 + |x_1|^2 \end{pmatrix} \geq 0.$$

It follows from Proposition 2.6 that all the principal minor determinants with order less than 4 of matrix $M(x)$ are semi-positive definite. So, to prove the positivity of $M(x)$, we need only to show that $\det(M(x)) \geq 0$. Writing $x_i = r_i e^{i\theta_i}$, $i = 1, 2, 3, 4$, we have

$$M(x) = U \begin{pmatrix} 2r_1^2 + r_2^2 & -r_1 r_2 & -r_1 r_3 & -r_1 r_4 \\ -r_1 r_2 & 2r_2^2 + r_3^2 & -r_2 r_3 & -r_2 r_4 \\ -r_1 r_3 & -r_2 r_3 & 2r_3^2 + r_4^2 & -r_3 r_4 \\ -r_1 r_4 & -r_2 r_4 & -r_3 r_4 & 2r_4^2 + r_1^2 \end{pmatrix} U^\dagger,$$

where

$$U = \begin{pmatrix} e^{i\theta_1} & 0 & 0 & 0 \\ 0 & e^{i\theta_2} & 0 & 0 \\ 0 & 0 & e^{i\theta_3} & 0 \\ 0 & 0 & 0 & e^{i\theta_4} \end{pmatrix}$$

is a unitary matrix. It follows that Φ is positive if and only if the determinant

$$f(r_1, r_2, r_3, r_4) = \begin{vmatrix} 2r_1^2 + r_2^2 & -r_1 r_2 & -r_1 r_3 & -r_1 r_4 \\ -r_1 r_2 & 2r_2^2 + r_3^2 & -r_2 r_3 & -r_2 r_4 \\ -r_1 r_3 & -r_2 r_3 & 2r_3^2 + r_4^2 & -r_3 r_4 \\ -r_1 r_4 & -r_2 r_4 & -r_3 r_4 & 2r_4^2 + r_1^2 \end{vmatrix} \geq 0$$

holds for all $0 \leq r_1, r_2, r_3, r_4 \leq 1$ with $r_1^2 + r_2^2 + r_3^2 + r_4^2 = 1$. This is the case since, by a computation, $\min f(r_1, r_2, r_3, r_4) = 0$ (also, refer to the proof of Theorem 5.1). So Φ is positive, as desired.

Similarly, one can show that Φ' and Φ'' are positive but not completely positive. The fact that Φ and Φ'' are indecomposable will be illustrated by Example 4.2 or 4.3 below. \square

Now let us give some examples.

The entanglement of the states ρ in Example 4.2 were studied in [8] by constructing suitable witnesses. We detect them by the positive maps obtained in Theorem 4.1. In addition, we also discuss the question when these states are entangled but cannot be recognized by the PPT criterion and the realignment criterion.

Example 4.2. Let H and K be Hilbert spaces of dimension ≥ 4 , and let $\{|i\rangle\}_{i=1}^4$ and $\{|j'\rangle\}_{j=1}^4$ be any orthonormal sets of H and K , respectively. Let $|\omega\rangle = \frac{1}{2}(|11'\rangle + |22'\rangle + |33'\rangle + |44'\rangle)$. Define $\rho_1 = |\omega\rangle\langle\omega|$, $\rho_2 = \frac{1}{4}(|12'\rangle\langle 12'| + |23'\rangle\langle 23'| + |34'\rangle\langle 34'| + |41'\rangle\langle 41'|)$, $\rho_3 = \frac{1}{4}(|13'\rangle\langle 13'| + |24'\rangle\langle 24'| + |31'\rangle\langle 31'| + |42'\rangle\langle 42'|)$ and $\rho_4 = \frac{1}{4}(|14'\rangle\langle 14'| + |21'\rangle\langle 21'| + |32'\rangle\langle 32'| + |43'\rangle\langle 43'|)$. Let $\rho = \sum_{i=1}^4 q_i \rho_i$ and $\rho_t = (1-t)\rho + t\rho_0$, where $q_i \geq 0$ for $i = 1, 2, 3, 4$ with $q_1 + q_2 + q_3 + q_4 = 1$, $t \in [0, 1]$, and ρ_0 is a state on $H \otimes K$. Then for sufficiently small t , or for ρ_0 with $(\Phi \otimes I)\rho_0 = (\Phi' \otimes I)\rho_0 = (\Phi'' \otimes I)\rho_0 = 0$, the following statements are true.

- (1) If $q_i < q_1$ for some $i = 2, 3, 4$, then ρ_t is entangled.
- (2) Let ρ_0 be PPT. Then ρ_t is PPT if and only if $q_2q_4 \geq q_1^2$ and $q_3 \geq q_1^2$. Thus, if $0 < q_i < q_1 < \frac{1}{4}$, $\frac{1}{4} \leq q_j < 1$ with $q_iq_j \geq q_1^2$ and $0 < q_1 \leq q_3 < 1$, where $i, j \in \{2, 4\}$ and $i \neq j$, then ρ_t is PPT entangled;
- (3) if ρ_0 is PPT, and if $q_1 \leq \frac{1}{7}$, $q_i = \frac{1}{2}q_1$, $q_j = \frac{1}{2}$ and $q_3 = \frac{1}{2} - 3q_i$, where $i, j \in \{2, 4\}$ and $i \neq j$, then ρ_t is PPT entangled but can not be detected by the realignment criterion.

We need only check ρ .

Denote by $F_{k,l}$ the unit matrix with (k, l) -entry 1 and others 0. For $\rho = \sum_{i=1}^4 q_i \rho_i$, we have

$$\begin{aligned} \rho = & \frac{1}{4} \text{diag}(q_1, q_4, q_3, q_2, q_2, q_1, q_4, q_3, q_3, q_2, q_1, q_4, q_4, q_3, q_2, q_1) \\ & + \frac{q_1}{4} (F_{1,6} + F_{1,11} + F_{1,16} + F_{6,1} + F_{6,11} + F_{6,16} \\ & + F_{11,1} + F_{11,6} + F_{11,16} + F_{16,1} + F_{16,6} + F_{16,11}) \end{aligned}$$

and

$$\begin{aligned} & 4(\Phi \otimes I)(\rho) \\ = & \text{diag}(2q_1 + q_4, 2q_4 + q_3, 2q_3 + q_2, 2q_2 + q_1, 2q_2 + q_1, 2q_1 + q_4, 2q_4 + q_3, \\ & 2q_3 + q_2, 2q_3 + q_2, 2q_2 + q_1, 2q_1 + q_4, 2q_4 + q_3, 2q_4 + q_3, 2q_3 + q_2, 2q_2 + q_1, 2q_1 + q_4) \\ & - q_1 (F_{1,6} + F_{1,11} + F_{1,16} + F_{6,1} + F_{6,11} + F_{6,16} \\ & + F_{11,1} + F_{11,6} + F_{11,16} + F_{16,1} + F_{16,6} + F_{16,11}), \end{aligned}$$

which is unitarily equivalent to

$$\begin{aligned} & \begin{pmatrix} 2q_1 + q_4 & -q_1 & -q_1 & -q_1 \\ -q_1 & 2q_1 + q_4 & -q_1 & -q_1 \\ -q_1 & -q_1 & 2q_1 + q_4 & -q_1 \\ -q_1 & -q_1 & -q_1 & 2q_1 + q_4 \end{pmatrix} \oplus (2q_4 + q_3)I_4 \\ & \oplus (2q_3 + q_2)I_4 \oplus (2q_2 + q_1)I_4 \oplus 0. \end{aligned}$$

Hence, by Proposition 2.6, we get that $(\Phi \otimes I)(\rho) \not\geq 0$ if $q_4 < q_1$, which implies that ρ is entangled if $q_4 < q_1$.

Note that

$$\rho \text{ is PPT if and only if } q_2q_4 \geq q_1^2 \text{ and } q_3 \geq q_1. \quad (4.2)$$

Thus we obtain that ρ is PPT entangled if $0 < q_4 < q_1 < \frac{1}{4}$, $\frac{1}{4} \leq q_2 < 1$ with $q_2q_4 \geq q_1^2$ and $0 < q_1 \leq q_3 < 1$. This reveals that the positive map Φ can recognize some PPT entangled states and hence is not decomposable.

The realignment matrix of ρ is

$$\begin{aligned} \rho^R &\cong \frac{1}{4} \text{diag}(q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1) \\ &+ \frac{q_4}{4} (F_{1,6} + F_{6,11} + F_{11,16} + F_{16,1}) + \frac{q_3}{4} (F_{1,11} + F_{6,16} + F_{11,1} + F_{16,6}) \\ &+ \frac{q_2}{4} (F_{1,16} + F_{6,1} + F_{11,6} + F_{16,11}) \\ &\cong \frac{1}{4} \begin{pmatrix} q_1 & q_4 & q_3 & q_2 \\ q_2 & q_1 & q_4 & q_3 \\ q_3 & q_2 & q_1 & q_4 \\ q_4 & q_3 & q_2 & q_1 \end{pmatrix} \oplus \frac{1}{4} q_1 I_{12} \oplus 0 = A \oplus \frac{1}{4} q_1 I_{12} \oplus 0. \end{aligned}$$

Thus $\|\rho^R\|_1 = \|A\|_1 + 3q_1$. By computation, we have that

$$\begin{aligned} \|A\|_1 &= \frac{3}{4} \sqrt{\sum_{i=1}^4 q_i^2 - q_1 q_2 - q_2 q_3 - q_3 q_4 - q_1 q_4} \\ &+ \frac{1}{4} \sqrt{\sum_{i=1}^4 q_i^2 + 3(q_1 q_2 + q_2 q_3 + q_3 q_4 + q_1 q_4)}. \end{aligned} \quad (4.3)$$

it follows from Eqs.(4.2)-(4.3) that the PPT criterion and the realignment criterion are independent each other. It is also easy to construct entangled states that can not be recognized by the PPT criterion and the realignment criterion. In fact, we have that $\|\rho^R\|_1 < 1$ if $q_1 \leq \frac{1}{7}$, $q_4 = \frac{1}{2}q_1$, $q_2 = \frac{1}{2}$ and $q_3 = \frac{1}{2} - 3q_4$. For example, $\|\rho^R\|_1 \doteq 0.9411 < 1$ if $q_1 = \frac{1}{7}$, $q_4 = \frac{1}{14}$, $q_2 = \frac{1}{2}$ and $q_3 = \frac{2}{7}$. Hence, in this case, the state ρ is PPT and cannot be detected by the realignment criterion. However it is entangled and can be recognized by the positive map Φ in Theorem 4.1.

Similarly, by applying the map Φ'' , we have that ρ is entangled if $q_2 < q_1$, and, ρ is PPT entangled if $0 < q_2 < q_1 < \frac{1}{4}$, $\frac{1}{4} \leq q_4 < 1$ with $q_2 q_4 \geq q_1^2$ and $0 < q_1 \leq q_3 < 1$. Thus, Φ'' is indecomposable, too. Furthermore, if $q_1 \leq \frac{1}{7}$, $q_2 = \frac{1}{2}q_1$, $q_4 = \frac{1}{2}$ and $q_3 = \frac{1}{2} - 3q_2$, then ρ is PPT entangled that cannot be detected by the realignment criterion. However, it can be detected by the positive map Φ'' in Theorem 4.1.

By applying the map Φ' , we see that ρ is entangled if $q_3 < q_1$. However, one should be careful that, in this case, ρ is not PPT. This means that we can not use ρ to check whether or not Φ' is decomposable.

The following example is new.

Example 4.3. Let H and K be complex Hilbert spaces of dimension ≥ 4 and let $\{|i\rangle\}_{i=1}^{\dim H}$ and $\{|j'\rangle\}_{j=1}^{\dim K}$ be any orthonormal bases of H and K , respectively. Let

$$|\omega_1\rangle = \frac{1}{2}(|11'\rangle + |22'\rangle + |33'\rangle + |44'\rangle) \quad \text{and} \quad |\omega_2\rangle = \frac{1}{2}(|12'\rangle + |23'\rangle + |34'\rangle + |41'\rangle).$$

Define $\rho_1 = |\omega_1\rangle\langle\omega_1|$, $\rho_2 = |\omega_2\rangle\langle\omega_2|$, $\rho_3 = \frac{1}{4}(|13'\rangle\langle 13'| + |24'\rangle\langle 24'| + |31'\rangle\langle 31'| + |42'\rangle\langle 42'|)$ and $\rho_4 = \frac{1}{4}(|14'\rangle\langle 14'| + |21'\rangle\langle 21'| + |32'\rangle\langle 32'| + |43'\rangle\langle 43'|)$. Let $\rho = \sum_{i=1}^4 q_i \rho_i$ and $\rho_t = (1-t)\rho + t\rho_0$, where $q_i \geq 0$ for $i = 1, 2, 3, 4$ with $q_1 + q_2 + q_3 + q_4 = 1$, $t \in [0, 1]$, and ρ_0 is a state on $H \otimes K$.

By using of the positive finite rank elementary operators Φ , Φ' and Φ'' in Theorem 4.1, we get that, for sufficient small t or for any ρ_0 with $(\Phi \otimes I)\rho_0 = (\Phi' \otimes I)\rho_0 = (\Phi'' \otimes I)\rho_0 = 0$, the followings are true.

- (1) If $q_1 \neq q_2$ or $q_1 = q_2 > q_i$ for some $i \in \{3, 4\}$, then ρ_t is entangled.
- (2) Let ρ_0 be PPT. Then ρ_t is PPT if and only if $q_1(q_1q_3^2 - q_2^2q_3 - q_1^3) \geq q_2^2(q_1q_3 - q_2^2) \geq 0$ and $q_2(q_2q_4^2 - q_1^2q_4 - q_2^3) \geq q_1^2(q_2q_4 - q_1^2) \geq 0$. Hence, if, in addition, $q_1 \neq q_2$ or $q_1 = q_2 > q_i$ for some $i \in \{3, 4\}$, then ρ_t is PPT entangled.
- (3) If ρ_0 is separable, and if $\frac{1}{2}q_i = q_j \leq \frac{1}{15}$ and $q_3 = q_4$, where $i, j \in \{1, 2\}$ and $i \neq j$, then ρ_t is PPT entangled that cannot be detected by the realignment criterion.

We need only deal with ρ .

For $\rho = q_1\rho_1 + q_2\rho_2 + q_3\rho_3 + 4\rho_4$, it is obvious that

$$\begin{aligned} \rho = & \frac{q_1}{4}(F_{1,1} + F_{1,6} + F_{1,11} + F_{1,16} + F_{6,1} + F_{6,6} + F_{6,11} + F_{6,16} \\ & + F_{11,1} + F_{11,6} + F_{11,11} + F_{11,16} + F_{16,1} + F_{16,6} + F_{16,11} + F_{16,16}) \\ & + \frac{q_2}{4}(F_{4,4} + F_{4,5} + F_{4,10} + F_{4,15} + F_{5,4} + F_{5,5} + F_{5,10} + F_{5,15} \\ & + F_{10,4} + F_{10,5} + F_{10,10} + F_{10,15} + F_{15,4} + F_{15,5} + F_{15,10} + F_{15,15}) \\ & + \frac{q_3}{4}(F_{3,3} + F_{8,8} + F_{9,9} + F_{14,14}) + \frac{q_4}{4}(F_{2,2} + F_{7,7} + F_{12,12} + F_{13,13}). \end{aligned}$$

Note that

$$\begin{aligned} 4(\Phi \otimes I)(\rho) = & \text{diag}(2q_1 + q_4, q_3 + 2q_4, q_2 + 2q_3, q_1 + 2q_2, q_1 + 2q_2, 2q_1 + q_4, q_3 + 2q_4, q_2 + 2q_3, \\ & q_2 + 2q_3, q_1 + 2q_2, 2q_1 + q_4, q_3 + 2q_4, q_3 + 2q_4, q_2 + 2q_3, q_1 + 2q_2, 2q_1 + q_4) \\ & - q_1(F_{1,6} + F_{1,11} + F_{1,16} + F_{6,1} + F_{6,11} + F_{6,16} \\ & + F_{11,1} + F_{11,6} + F_{11,16} + F_{16,1} + F_{16,6} + F_{16,11}) \\ & - q_2(F_{4,5} + F_{4,10} + F_{4,15} + F_{5,4} + F_{5,10} + F_{5,15} \\ & + F_{10,4} + F_{10,5} + F_{10,15} + F_{15,4} + F_{15,5} + F_{15,10}), \end{aligned}$$

which is unitarily equivalent to the operator $A \oplus B \oplus C \oplus D \oplus 0$, where

$$A = \begin{pmatrix} 2q_1 + q_4 & -q_1 & -q_1 & -q_1 \\ -q_1 & 2q_1 + q_4 & -q_1 & -q_1 \\ -q_1 & -q_1 & 2q_1 + q_4 & -q_1 \\ -q_1 & -q_1 & -q_1 & 2q_1 + q_4 \end{pmatrix}, B = \begin{pmatrix} q_1 + 2q_2 & -q_2 & -q_2 & -q_2 \\ -q_2 & q_1 + 2q_2 & -q_2 & -q_2 \\ -q_2 & -q_2 & q_1 + 2q_2 & -q_2 \\ -q_2 & -q_2 & -q_2 & q_1 + 2q_2 \end{pmatrix}$$

and

$$C = \begin{pmatrix} q_2 + 2q_3 & 0 & 0 & 0 \\ 0 & q_2 + 2q_3 & 0 & 0 \\ 0 & 0 & q_2 + 2q_3 & 0 \\ 0 & 0 & 0 & q_2 + 2q_3 \end{pmatrix}, D = \begin{pmatrix} q_3 + 2q_4 & 0 & 0 & 0 \\ 0 & q_3 + 2q_4 & 0 & 0 \\ 0 & 0 & q_3 + 2q_4 & 0 \\ 0 & 0 & 0 & q_3 + 2q_4 \end{pmatrix}.$$

It is clear that $C, D \geq 0$. For the matrices A and B , by Proposition 2.6, we get that $A \geq 0$ if and only if $q_4 \geq q_1$ and $B \geq 0$ if and only if $q_1 \geq q_2$. So $(\Phi \otimes I)(\rho)$ is not positive if $q_4 < q_1$ or $q_1 < q_2$. It follows from the elementary operator criterion that ρ is entangled if $q_4 < q_1$ or $q_1 < q_2$.

Next, consider the positive partial transpose of ρ . It is clear that

$$\begin{aligned} \rho^{T_1} &\cong \frac{q_1}{4}(F_{1,1} + F_{2,5} + F_{3,9} + F_{4,13} + F_{5,2} + F_{6,6} + F_{7,10} + F_{8,14}) \\ &\quad + F_{9,3} + F_{10,7} + F_{11,11} + F_{12,15} + F_{13,4} + F_{14,8} + F_{15,12} + F_{16,16}) \\ &\quad + \frac{q_2}{4}(F_{1,8} + F_{2,12} + F_{3,16} + F_{4,4} + F_{5,5} + F_{6,9} + F_{7,13} + F_{8,1} \\ &\quad + F_{9,6} + F_{10,10} + F_{11,14} + F_{12,2} + F_{13,7} + F_{14,11} + F_{15,15} + F_{16,3}) \\ &\quad + \frac{q_3}{4}(F_{3,3} + F_{8,8} + F_{9,9} + F_{14,14}) + \frac{q_4}{4}(F_{2,2} + F_{7,7} + F_{12,12} + F_{13,13}) \\ &\cong A_1 \oplus B_1 \oplus C_1 \oplus D_1 \oplus 0, \end{aligned}$$

where

$$A_1 = \frac{1}{4} \begin{pmatrix} q_1 & q_2 & 0 & 0 \\ q_2 & q_3 & 0 & q_1 \\ 0 & 0 & q_1 & q_2 \\ 0 & q_1 & q_2 & q_3 \end{pmatrix}, \quad B_1 = \frac{1}{4} \begin{pmatrix} q_4 & q_1 & q_2 & 0 \\ q_1 & q_2 & 0 & 0 \\ q_2 & 0 & q_4 & q_1 \\ 0 & 0 & q_1 & q_2 \end{pmatrix}$$

and

$$C_1 = \frac{1}{4} \begin{pmatrix} q_3 & 0 & q_1 & q_2 \\ 0 & q_1 & q_2 & 0 \\ q_1 & q_2 & q_3 & 0 \\ q_2 & 0 & 0 & q_1 \end{pmatrix}, \quad D_1 = \frac{1}{4} \begin{pmatrix} q_2 & 0 & 0 & q_1 \\ 0 & q_4 & q_1 & q_2 \\ 0 & q_1 & q_2 & 0 \\ q_1 & q_2 & 0 & q_4 \end{pmatrix}.$$

It is easy to check that $A_1 \geq 0$ if and only if $q_1 q_3 \geq q_2^2$ and $q_1^2 q_3^2 - 2q_1 q_2^2 q_3 - q_1^4 + q_2^4 \geq 0$; $B_1 \geq 0$ if and only if $q_2 q_4 \geq q_1^2$ and $q_2^2 q_4^2 - 2q_1^2 q_2 q_4 + q_1^4 - q_2^4 \geq 0$; $C_1 \geq 0$ if and only if $q_1 q_3^2 \geq q_2^2 q_3 + q_1^3$ and $q_1^2 q_3^2 - 2q_1 q_2^2 q_3 - q_1^4 + q_2^4 \geq 0$; and $D_1 \geq 0$ if and only if $q_2 q_4 \geq q_1^2$ and $q_2^2 q_4^2 - 2q_1^2 q_2 q_4 + q_1^4 - q_2^4 \geq 0$. Hence

$$\begin{aligned} \rho \text{ is PPT if and only if} \\ q_1(q_1 q_3^2 - q_2^2 q_3 - q_1^3) \geq q_2^2(q_1 q_3 - q_2^2) \geq 0 \\ \text{and } q_2(q_2 q_4^2 - q_1^2 q_4 - q_2^3) \geq q_1^2(q_2 q_4 - q_1^2) \geq 0. \end{aligned} \tag{4.4}$$

Particularly,

$$\text{if } q_2 = 2q_1 \text{ and } q_3 = q_4 \geq 4q_1, \text{ then } \rho \text{ is PPT entangled.} \tag{4.5}$$

This fact will be used below.

Now, let us apply the realignment criterion to ρ . The realignment of ρ is

$$\begin{aligned}\rho^R &\cong \frac{1}{4}\text{diag}(q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1, q_1) \\ &\quad + \frac{q_2}{4}(F_{1,16} + F_{2,13} + F_{3,14} + F_{4,15} + F_{5,4} + F_{6,1} + F_{7,2} + F_{8,3} \\ &\quad + F_{9,8} + F_{10,5} + F_{11,6} + F_{12,7} + F_{13,12} + F_{14,9} + F_{15,10} + F_{16,11}) \\ &\quad + \frac{q_3}{4}(F_{1,11} + F_{6,16} + F_{11,1} + F_{16,6}) + \frac{q_4}{4}(F_{1,6} + F_{6,11} + F_{11,16} + F_{16,1}) \\ &\cong A \oplus B^{(3)} \oplus 0,\end{aligned}$$

where

$$A = \frac{1}{4} \begin{pmatrix} q_1 & q_4 & q_3 & q_2 \\ q_2 & q_1 & q_4 & q_3 \\ q_3 & q_2 & q_1 & q_4 \\ q_4 & q_3 & q_2 & q_1 \end{pmatrix}, \quad B = \frac{1}{4} \begin{pmatrix} q_1 & 0 & 0 & q_2 \\ q_2 & q_1 & 0 & 0 \\ 0 & q_2 & q_1 & 0 \\ 0 & 0 & q_2 & q_1 \end{pmatrix}$$

and $B^{(3)}$ denotes the direct sum of 3 copies of B . Then

$$\begin{aligned}\|\rho^R\|_1 &= \|A\|_1 + 3\|B\|_1 \\ &= \frac{3}{4}\sqrt{\sum_{i=1}^4 q_i^2 - q_1q_2 - q_2q_3 - q_3q_4 - q_1q_4} \\ &\quad + \frac{1}{4}\sqrt{\sum_{i=1}^4 q_i^2 + 3(q_1q_2 + q_2q_3 + q_3q_4 + q_1q_4)} \\ &\quad + \frac{9}{4}\sqrt{q_1^2 + q_2^2 - q_1q_2} + \frac{3}{4}\sqrt{q_1^2 + q_2^2 + 3q_1q_2}.\end{aligned}\tag{4.6}$$

Now a computation reveals that, if $q_1 \leq \frac{1}{15}$, $q_2 = 2q_1$ and $q_3 = q_4$, then the trace norm $\|\rho^R\|_1 < 1$. Note that, by Eq.(4.5), ρ is PPT in this case. Hence, we get another kind of examples of entangled states that are PPT and cannot be detected by the realignment criterion.

Similarly, by using the positive map Φ'' , we obtain that ρ is entangled if $q_2 < q_1$ or $q_3 < q_2$, and, if $q_2 \leq \frac{1}{15}$, $q_1 = 2q_2$ and $q_3 = q_4$, then ρ is PPT entangled that cannot be detected by the realignment criterion.

By using the positive map Φ' , we see that ρ is entangled if $q_3 < q_1$ or $q_4 < q_2$. In this case, by Eq.(4.4), ρ is not PPT because $q_1q_3^2 - q_2^2q_3 - q_1^3 < 0$ or $q_2q_4^2 - q_1^2q_4 - q_2^3 < 0$.

5. POSITIVE FINITE RANK ELEMENTARY OPERATORS OF ORDER (n, n)

In this section we consider the general case, that is, constructing positive finite rank elementary operators of order (n, n) . The main purpose is to show that the following result is true.

Theorem 5.1. *Let H and K be Hilbert spaces of dimension $\geq n$, and let $\{|i\rangle\}_{i=1}^n$ and $\{|j'\rangle\}_{j=1}^n$ be any orthonormal sets of H and K , respectively. For $k = 1, 2, \dots, n-1$, let*

$\Phi^{(k)} : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ be defined by

$$\begin{aligned} \Phi^{(k)}(A) = & (n-1) \sum_{i=1}^n E_{ii} A E_{ii}^\dagger + \sum_{i=1}^n E_{i, \pi^k(i)} A E_{i, \pi^k(i)}^\dagger \\ & - (\sum_{i=1}^n E_{ii}) A (\sum_{i=1}^n E_{ii})^\dagger \end{aligned} \quad (5.1)$$

for every $A \in \mathcal{B}(H)$, where $\pi(i) = \pi^1(i) = (i+1) \bmod n$, $\pi^k(i) = (i+k) \bmod n$ ($k > 1$), $i = 1, 2, \dots, n$ and $E_{ji} = |j'\rangle\langle i|$. Then $\Phi^{(k)}$ are positive but not completely positive. Moreover, $\Phi^{(k)}$ is indecomposable whenever either n is odd or $k \neq \frac{n}{2}$.

Proof. Obviously, $\Phi^{(k)}$ is not completely positive for each $k = 1, 2, \dots, n-1$. Similar to the proof of Theorem 4.1, to prove that $\Phi = \Phi^{(1)}$ is positive, it is sufficient to show that the function

$$= \left| \begin{array}{ccccc} f_{1,n}(r_1, r_2, \dots, r_n) & & & & \\ (n-2)r_1^2 + r_2^2 & -r_1r_2 & -r_1r_3 & \cdots & -r_1r_n \\ -r_1r_2 & (n-2)r_2^2 + r_3^2 & -r_2r_3 & \cdots & -r_2r_n \\ -r_1r_3 & -r_2r_3 & (n-2)r_3^2 + r_4^2 & \cdots & -r_3r_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -r_1r_n & -r_2r_n & -r_3r_n & \cdots & (n-2)r_n^2 + r_1^2 \end{array} \right| \geq 0 \quad (5.2)$$

for all (r_1, r_2, \dots, r_n) with $0 \leq r_1, r_2, \dots, r_n \leq 1$ and $\sum_{i=1}^n r_i^2 = 1$. Other $\Phi^{(k)}$ s are dealt with similarly.

We may assume that all r_i s are nonzero. Let $x_i = \frac{r_{i+1}^2}{r_i^2}$, $i = 1, 2, \dots, n-1$, and $x_n = \frac{r_1^2}{r_n^2}$. Then $x_1 x_2 \cdots x_n = 1$ and

$$f_{1,n}(r_1, r_2, \dots, r_n) = (r_1 r_2 \cdots r_n)^2 h_{1,n}(x_1, x_2, \dots, x_n), \quad (5.3)$$

where

$$= \left| \begin{array}{ccccc} h_{1,n}(x_1, x_2, \dots, x_n) & & & & \\ (n-2) + x_1 & -1 & -1 & \cdots & -1 \\ -1 & (n-2) + x_2 & -1 & \cdots & -1 \\ -1 & -1 & (n-2) + x_3 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & (n-2) + x_n \end{array} \right| \quad (5.4)$$

with each $x_i > 0$ and $x_1 x_2 \cdots x_n = 1$. It follows that $f_{1,n} \geq 0$ for all (r_1, r_2, \dots, r_n) with $0 \leq r_1, r_2, \dots, r_n \leq 1$ and $\sum_{i=1}^n r_i^2 = 1$ if and only if $h_{1,n} \geq 0$ holds for all (x_1, x_2, \dots, x_n) with $x_i > 0$ ($i = 1, 2, \dots, n$) and $x_1 x_2 \cdots x_n = 1$.

Note that, the determinant in Eq.(5.4) can be formulated as

$$\begin{aligned} h_{1,n}(x_1, x_2, \dots, x_n) = & -M_0 + M_1 \sum_{i=1}^n x_i + M_2 \sum_{i < j} x_i x_j + \dots \\ & + M_k \sum_{i_1 < i_2 < \dots < i_k} x_{i_1} x_{i_2} \dots x_{i_k} + \dots \\ & + M_{n-1} \sum_{i_1 < i_2 < \dots < i_{n-1}} x_{i_1} x_{i_2} \dots x_{i_{n-1}} + M_n x_1 x_2 \dots x_n. \end{aligned}$$

The case of $n = 3$ is obvious. So we assume that $n \geq 4$ in the sequel. Since, by Proposition 2.6, $h_{1,n}(0, 0, \dots, 0) = -M_0 < 0$, we have $M_0 > 0$. By taking $x_i = 0$ for $2 \leq i \leq n$, it is easily checked that $M_1 = h_{1,n-1}(1, 1, \dots, 1)$. Let $x_i = 0$ for $i \geq 3$. A computation reveals that $M_2 = h_{1,n-2}(2, 2, \dots, 2) \geq 0$. In general, one can check that

$$M_k = h_{1,n-k}(k, k, \dots, k) \geq 0, \quad k = 1, 2, \dots, n. \quad (5.5)$$

For example,

$$M_{n-3} = h_{1,3}(n-3, n-3, n-3) = \begin{vmatrix} n-2 & -1 & -1 \\ -1 & n-2 & -1 \\ -1 & -1 & n-2 \end{vmatrix} = (n-2)^3 - 3(n-2) - 2 \geq 0,$$

$$M_{n-2} = h_{1,2}(n-2, n-2) = \begin{vmatrix} n-2 & -1 \\ -1 & n-2 \end{vmatrix} = (n-2)^2 - 1 \geq 0,$$

$M_{n-1} = h_{1,n-1} = n-2 \geq 0$ and $M_n = 1$. Thus we have shown that $M_0, M_1, M_2, \dots, M_n \in \mathbb{N} \cup \{0\}$. It is easily checked that $h_{1,n}(1, 1, \dots, 1) = 0$. This leads to

$$\sum_{i=1}^n M_i = M_0. \quad (5.6)$$

Next, observe that if $a_j > 0$ and $a_1 a_2 \dots a_m = 1$, then $\sum_{j=1}^m a_j \geq 1$. It follows that

$$\sum_{i_1 < i_2 < \dots < i_k} x_{i_1} x_{i_2} \dots x_{i_k} \geq 1 \quad (5.7)$$

holds for each $1 \leq k \leq n$. Eq.(5.7), together with Eq.(5.6), yields that $h_{1,n}(x_1, x_2, \dots, x_n) \geq 0$ holds for all (x_1, x_2, \dots, x_n) with $x_1 x_2 \dots x_n = 1$.

The last assertion will be proved by Example 5.4 below. The proof is finished. \square

Remark 5.2. Let π be any permutation of $(1, 2, \dots, n)$ and let $\Psi_\pi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be the map defined by

$$\Psi_\pi(A) = \text{diag}\{(n-1)a_{11} + a_{\pi(1)\pi(1)}, (n-1)a_{22} + a_{\pi(2)\pi(2)}, \dots, (n-1)a_{nn} + a_{\pi(n)\pi(n)}\} - A$$

for every $A = (a_{ij}) \in M_n(\mathbb{C})$. By Theorem 2.1, Proposition 2.7 and the proof of Theorem 5.1, it is easily seen that Ψ_π is a positive linear map that is not completely positive whenever $\pi \neq \text{id}$.

Remark 5.3. For any n -dimensional Hilbert space H , define

$$J_k = \frac{1}{\sqrt{k(k+1)}} \left(\sum_{i=1}^{k-1} E_{ii} - (k-1)E_{kk} \right), \quad k = 1, 2, \dots, n-1,$$

$$J_s = \begin{cases} \frac{1}{\sqrt{2}}(E_{ij} + E_{ji}), & \text{if } k \text{ is odd and } i < j, \\ \frac{1}{\sqrt{2}}(iE_{ij} - iE_{ji}), & \text{if } k \text{ is even and } i < j. \end{cases}$$

Relabel these $n^2 - 1$ matrices as $J_1, J_2, \dots, J_{n^2-1}$. Then the $n^2 - 1$ matrices form a completely orthonormal traceless set and any $n \times n$ Hermitian matrix S can be written as the form

$$S = \frac{1}{n} \left(I + \sum_{k=1}^{n^2-1} \eta_k J_k \right),$$

where $\eta_k \in \mathbb{R}$, $k = 1, 2, \dots, n^2 - 1$. Hence it is clear that the $n \times n$ hermitian matrices with trace 1 and the points in \mathbb{R}^{n^2-1} (the real linear space) are in one-to-one correspondence. The image Λ_n of the set of all density matrices is a closed convex set in \mathbb{R}^{n^2-1} . Then every positive linear map $\Phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ corresponds to a linear map $M_\Phi : \mathbb{R}^{n^2-1} \rightarrow \mathbb{R}^{n^2-1}$ that sends Λ_n into Λ_n . It was shown in [15] that every map represented by a matrix of the form $M = (n-1)^{-1}R$ is positive, where $R \in \mathcal{O}(n^2-1)$, the orthogonal group of proper and improper rotations in \mathbb{R}^{n^2-1} ([15, Theorem 4]). Some more can be said. In fact, $M = (n-1)^{-1}R$ corresponds a positive map whenever $\|R\| \leq 1$. The positive maps in Theorem 3.1 may be obtained from this way. However, the positive maps in Theorem 4.1 can not be obtained from this way. For example, consider the map Φ in Theorem 4.1. By a simple calculation, we get

$$M_\Phi = \frac{1}{18} \begin{pmatrix} 9 & 3\sqrt{3} & 0 \\ -\sqrt{3} & 11 & 4\sqrt{2} \\ -2\sqrt{6} & -2\sqrt{2} & 10 \end{pmatrix}.$$

It is clear that $\|M\| > \frac{1}{3}$, and so [15, Theorem 4] is not applicable to our map Φ here.

In the following we give two examples that generalize the examples in Sections 3-4.

The states ρ in Example 5.4 were suggested in [8] without analyzing their entanglement.

Example 5.4. Let H and K be Hilbert spaces of dimension $\geq n$ and let $\{|i\rangle\}_{i=1}^n$ and $\{|j'\rangle\}_{j=1}^n$ be any orthonormal sets of H and K , respectively. Let $|\omega\rangle = \frac{1}{n} \sum_{i=1}^n |ii'\rangle$. Define $\rho_1 = |\omega\rangle\langle\omega|$, $\rho_2 = \frac{1}{n} \sum_{i=1}^n (I \otimes S) |ii'\rangle\langle ii'| (I \otimes S)^\dagger$, $\rho_3 = \frac{1}{n} \sum_{i=1}^n (I \otimes S^2) |ii'\rangle\langle ii'| (I \otimes S^2)^\dagger$, \dots , $\rho_n = \frac{1}{n} \sum_{i=1}^n (I \otimes S^{n-1}) |ii'\rangle\langle ii'| (I \otimes S^{(n-1)})^\dagger$, where S is the operator on K defined by $S|j'\rangle = |(j+1)'\rangle$ if $j = 1, 2, \dots, n-1$, $S|n'\rangle = |1'\rangle$ and $S|j'\rangle = 0$ if $j > n$. Let $\rho = \sum_{i=1}^n q_i \rho_i$ and $\rho_t = (1-t)\rho + t\rho_0$, where $q_i \geq 0$ for $i = 1, 2, \dots, n$ with $\sum_{i=1}^n q_i = 1$, $t \in [0, 1]$, and ρ_0 is a state on $H \otimes K$. Then for sufficiently small t , or for ρ_0 with $(\Phi^{(k)} \otimes I)\rho_0 = 0$ $k = 1, 2, \dots, n-1$, the following statements are true.

- (1) If $q_i < q_1$ for some $i = 2, 3, \dots, n$, then ρ_t is entangled;
 (2) Let ρ_0 be PPT. Then ρ_t is a PPT state if and only if $q_i q_j \geq q_1^2$ for i, j with $i + j = n + 2$, $i = 3, 4, \dots, n$.

It is enough to discuss the entanglement of ρ . For $\rho = \sum_{i=1}^n q_i \rho_i$, by using the map $\Phi = \Phi^{(1)}$ in Theorem 5.1, it is easily checked that

$$\begin{aligned}
 & n(\Phi \otimes I)(\rho) \\
 & \cong \begin{pmatrix} (n-2)q_1 + q_n & -q_1 & -q_1 & \cdots & -q_1 \\ -q_1 & (n-2)q_1 + q_n & -q_1 & \cdots & -q_1 \\ -q_1 & -q_1 & (n-2)q_1 + q_n & \cdots & -q_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -q_1 & -q_1 & -q_1 & \cdots & (n-2)q_1 + q_n \end{pmatrix} \\
 & \oplus ((n-2)q_n + q_{n-1})I_n \oplus ((n-2)q_{n-1} + q_{n-2})I_n \oplus \cdots \oplus ((n-2)q_2 + q_1)I_n \oplus 0.
 \end{aligned}$$

Thus, by Proposition 2.6, we get that ρ is entangled if $q_n < q_1$.

Similarly, by applying the map $\Phi^{(k)}$ in Theorem 5.1, we have ρ is entangled if $q_{n+1-k} < q_1$, where $k = 2, 3, \dots, n-1$.

It is easily checked that ρ is PPT if and only if $q_i q_j \geq q_1^2$, where $i + j = n + 2$ and $i = 3, 4, \dots, n$.

Moreover, if n is odd, or if n is even but $k \neq \frac{n}{2}$, we can choose q_1, q_2, \dots, q_n so that $q_{n+1-k} < q_1 < \frac{1}{n}$ and $q_i q_j \geq q_i^2$ whenever $i + j = n + 2$. It follows that $\rho = \sum_{i=1}^n q_i \rho_i$ is PPT entangled which can be recognized by $\Phi^{(k)}$. Hence, $\Phi^{(k)}$ is not decomposable. This completes the proof of the last assertion of Theorem 5.1.

Example 5.5. Let H and K be complex Hilbert spaces of dimension $\geq n$ and let $\{|i\rangle\}_{i=1}^{\dim H}$ and $\{|j'\rangle\}_{j'=1}^{\dim K}$ be any orthonormal bases of H and K , respectively. Let $|\omega_1\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^n |ii'\rangle$ and $|\omega_2\rangle = \frac{1}{\sqrt{n}}(|12'\rangle + |23'\rangle + \cdots + |(n-1)n'\rangle + |n1'\rangle)$. Define $\rho_1 = |\omega_1\rangle\langle\omega_1|$, $\rho_2 = |\omega_2\rangle\langle\omega_2|$, $\rho_3 = \frac{1}{n} \sum_{i=1}^n (I \otimes S^2)|ii'\rangle\langle ii'|(I \otimes S^{2\dagger})$, \dots , $\rho_n = \frac{1}{n} \sum_{i=1}^n (I \otimes S^{n-1})|ii'\rangle\langle ii'|(I \otimes S^{(n-1)\dagger})$, where S is the same operator as in Example 5.4. Let $\rho = \sum_{i=1}^n q_i \rho_i$ and $\rho_t = (1-t)\rho + t\rho_0$, where $q_i \geq 0$ for $i = 1, 2, \dots, n$ with $\sum_{i=1}^n q_i = 1$, $t \in [0, 1]$, and ρ_0 is a state on $H \otimes K$. By using of the positive finite rank elementary operators $\Phi^{(k)}$ in Theorem 5.1, we can get that, for sufficient small t or for any ρ_0 with $(\Phi^{(k)} \otimes I)\rho_0 = 0$, $k = 1, 2, \dots, n-1$, if $q_1 \neq q_2$ or $q_1 = q_2 > q_i$ for some $i \in \{3, 4, \dots, n\}$, then ρ_t is entangled.

Still, we only need to consider the entanglement of ρ . For $\rho = \sum_{i=1}^n q_i \rho_i$, with $\Phi = \Phi^{(1)}$ as in Theorem 5.1, it is clear that

$$n(\Phi \otimes I)(\rho) \cong \begin{pmatrix} (n-2)q_1 + q_n & -q_1 & -q_1 & \cdots & -q_1 \\ -q_1 & (n-2)q_1 + q_n & -q_1 & \cdots & -q_1 \\ -q_1 & -q_1 & (n-2)q_1 + q_n & \cdots & -q_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -q_1 & -q_1 & -q_1 & \cdots & (n-2)q_1 + q_n \end{pmatrix} \oplus \begin{pmatrix} (n-2)q_2 + q_1 & -q_2 & -q_2 & \cdots & -q_2 \\ -q_2 & (n-2)q_2 + q_1 & -q_2 & \cdots & -q_2 \\ -q_2 & -q_2 & (n-2)q_2 + q_1 & \cdots & -q_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -q_2 & -q_2 & -q_2 & \cdots & (n-2)q_2 + q_1 \end{pmatrix} \oplus \bigoplus_{k=3}^n ((n-2)q_k + q_{k-1})I_n \oplus 0$$

So, by Proposition 2.6, $(\Phi \otimes I)(\rho)$ is not positive if $q_n < q_1$ or $q_1 < q_2$. It follows from the elementary operator criterion that ρ is entangled if $q_n < q_1$ or $q_1 < q_2$.

Similarly, by applying the map $\Phi^{(k)}$ ($k = 2, 3, \dots, n-1$) in Theorem 5.1, one gets that ρ is entangled if $q_{n+1-k} < q_1$ or $q_1 < q_2$. Thus, we obtain that ρ is entangled if $q_1 \neq q_2$ or $q_1 = q_2 > q_i$ for some $i \in \{3, 4, \dots, n\}$.

Before the end of this section, we propose a question.

Question 5.6. Let $n \geq 4$ be an even integer. Is the positive map $\Phi^{(\frac{n}{2})}$ defined in Theorem 5.1 indecomposable? Particularly, is the positive map Φ' defined in Theorem 4.1 indecomposable?

We guess that the answer is affirmative, but we are not able to prove it here.

6. CONCLUSIONS

Let H and K be complex Hilbert spaces of any dimension. By the elementary operator criterion [13], a state ρ on $H \otimes K$ is entangled if and only if there exists a positive finite rank elementary operator $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ that is not completely positive (NCP) such that $(\Phi \otimes I)\rho$ is not positive. Hence it is important and interesting to construct positive finite rank elementary operators that are NCP. In this paper, we construct some new positive finite rank elementary operators and apply them to get some new examples of entangled states. We also give a necessary and sufficient condition for a pure state to be separable in terms of a special positive elementary operator of order $(2, 2)$.

More concretely, for any positive integer $n \geq 3$, the NCP positive finite rank elementary operators that we constructed are $\Phi^{(k)} : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$ defined by $\Phi^{(k)}(A) = (n -$

1) $\sum_{i=1}^n E_{ii} A E_{ii}^\dagger + \sum_{i=1}^n E_{i, \pi^k(i)} A E_{i, \pi^k(i)}^\dagger - (\sum_{i=1}^n E_{ii}) A (\sum_{i=1}^n E_{ii})^\dagger$ for every $A \in \mathcal{B}(H)$, $k = 1, 2, \dots, n-1$, where $\{|i\rangle\}_{i=1}^n$ and $\{|j'\rangle\}_{j=1}^n$ are any orthonormal sets of H and K , respectively, $E_{ji} = |j'\rangle\langle i|$ and $\pi^1 = \pi$ is a permutation of $\{1, 2, \dots, n\}$ defined by $\pi(i) = (i+1) \bmod n$, $\pi^k(i) = (i+k) \bmod n$ ($k > 1$), $i = 1, 2, \dots, n$. Moreover, we show that $\Phi^{(k)}$ is indecomposable whenever either n is odd or n is even but $k \neq \frac{n}{2}$. We discuss two kinds of entangled states to illustrate how to use these positive maps to detect the entanglement of states. Especially, we study the examples in detail for the case $n = 4$ to determine when they are PPT and when they can be detected by the realignment criterion, and get some new examples of entangled states that cannot be recognized by the PPT criterion and the realignment criterion.

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